AFFINE PAVINGS FOR AFFINE SPRINGER FIBERS FOR SPLIT ELEMENTS IN $PGL(3)$

VINCENT LUCARELLI

Abstract. This paper constructs pavings by affine spaces for the affine Springer fibers for $PGL(3)$ obtained from regular compact elements in a split maximal torus. These pavings are constructed by intersecting the affine Springer fiber with a non-standard paving of the affine Grassmannian.

Let k be a field, $F = k((\pi))$ be the field of formal Laurent series over k, $\mathcal{O} = k[[\pi]]$ be the subring of formal power series, and $\mathcal{P} = \pi\mathcal{O}$ be the maximal ideal. Let $G = PGL_3(F)$ and $K = PGL_3(\mathcal{O})$. We write X for the affine Grassmannian G/K and A for the diagonal maximal torus in G.

In section 2.2 we develop for each integer $a \geq 0$ a non-standard paving of X by affine spaces, referred to as the a-paving of X. These non-standard pavings are constructed using the standard Iwahori subgroup I and a conjugate of it, I^a, that depends on a. Each affine space in an a-paving is a union of intersections of I-orbits and I^a-orbits in X. In particular, the affine spaces are preserved by $I \cap I^a$ and hence by its subgroup $A(\mathcal{O})$.

Each affine space in an a-paving contains exactly one element of $X_*(A)$ and thus these lattice points index the affine spaces. This is analogous to how affine spaces are indexed in the standard paving of X by I-orbits. In fact, when $a = 0$ the a-paving of X is identical to the standard affine paving of X by I-orbits.

An a-paving differs from the standard paving of X by I-orbits in how the closure of each affine space relates to the paving. In the standard paving, the closure of an affine space is the union of smaller dimensional affine spaces. For all a other than zero, our a-paving is a paving in a weaker sense. The closure of each affine space is not necessarily the union of other affine spaces in the paving. However, we can order the affine spaces $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \dots$ so that $\mathbb{A}_0 \cup \cdots \cup \mathbb{A}_n$ is closed for all n.

Our non-standard pavings are interesting because they induce affine pavings of certain fixed point sets in X. Specifically, when γ is a regular element in $A(\mathcal{O})$ the set

$$X^\gamma = \{g \in G/K : \gamma g = g\}$$

admits a paving by affine spaces when intersected with a particular a-paving of X that is determined by γ. The set X^γ is called an affine Springer fiber. They were first studied by Kazhdan and Lusztig in [KL88].

The value of a that induces an affine paving of X^γ is determined as follows. We use $v(x)$ to denote the valuation of $x \in F$ and we take $v(0) = +\infty$. Now γ has the

Date: September 5, 2003.
2000 *Mathematics Subject Classification.* 22E67; 14L40.

form

$$\gamma = \begin{bmatrix} u_1 & & \\ & u_2 & \\ & & u_3 \end{bmatrix}$$

where u_1, u_2, and u_3 are in $\mathcal{O}^\times$ and distinct. We can permute the entries of γ so that

$$v\left(1 - \frac{u_1}{u_2}\right) = v\left(1 - \frac{u_1}{u_3}\right) = m$$

and

$$v\left(1 - \frac{u_2}{u_3}\right) = n$$

with $n \geq m \geq 0$. Then $a = n - m$.

Main Theorem. *The intersection of X^γ with the a-paving of X determined by $a = n - m$ yields an affine paving of X^γ. In particular, the intersection of X^γ with each affine space in the a-paving of X is an affine space.*

To prove the result, we explicitly calculate the intersection of X^γ with the affine spaces of the a-paving of X in section 3. The precise statement of the main theorem is Theorem 1.1.

Many other authors have found affine pavings of affine Springer fibers for specific groups with an equivalued condition on the valuation of roots [Fan96] [LW] [LS91] [Sag00] [Som97]. Goresky, Kottwitz, and MacPherson proved a general result that gives an affine paving of affine Springer fibers for any connected reductive group assuming the equivalued condition [GKM]. This paper develops the first known affine paving of an affine Springer fiber in the non-equivalued case.

We conjecture that this method can be extended to prove the same result when k is algebraically closed, $char(k) \neq 2$, and the maximal torus splits as $E^\times \times F^\times$ where E/F is a quadratic extension. Provided this is true, in $PGL(3)$ an affine paving of X^γ is known for all regular semisimple γ except in characteristic 2 and 3. In the case of the torus $E^\times$ where E/F is a cubic extension, n and m are forced to be equal and in characteristic other than 2 and 3 [GKM] applies. ([GKM] actually applies to the Lie algebra of G, but the result is equivalent for the group.)

I wish to thank my advisor Robert Kottwitz for suggesting this problem and for his generous help.

1. Notation, Definitions, & Precise Statement of the Main Theorem

An element of X that has a diagonal matrix coset representative can be expressed in terms of a diagonal matrix with monomial entries. That representative is equivalent in G to an element of the form

$$\begin{bmatrix} 1 & & \\ & \pi^s & \\ & & \pi^t \end{bmatrix}.$$

We denote such elements of X by the coordinates (s, t). (These points are the vertices in the main apartment of the building for G.)

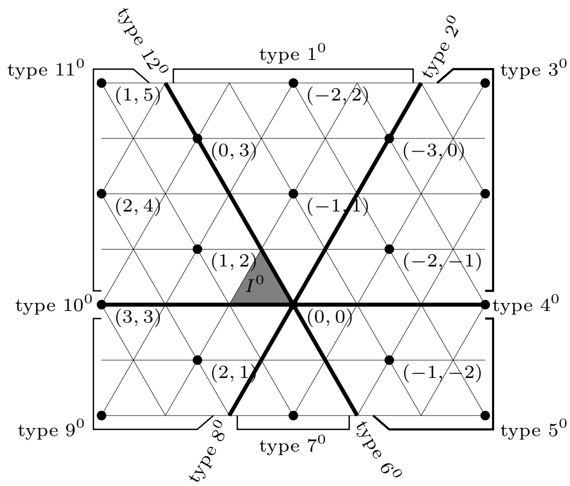

FIGURE 1. Example for $a = 0$

Let I denote the standard Iwahori subgroup

$$I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix}$$

and for $a \geq 0$ define the conjugate

$$\begin{aligned} I^a &= \begin{bmatrix} 1 & & \\ & \pi^a & \\ & & \pi^a \end{bmatrix} I \begin{bmatrix} 1 & & \\ & \pi^{-a} & \\ & & \pi^{-a} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-a} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix}. \end{aligned}$$

The subgroup I^a acts on points of X by left multiplication. This action fixes the point (a, a), which we call the base point relative to a. For a particular I^a, the other points (s, t) in the main apartment of X can be divided into 12 types based on the relationship of a and the coordinates (s, t).

Type	Condition
1^a	$s < a < t$
2^a	$s < t = a$
3^a	$s < t < a$
4^a	$s = t < a$

Type	Condition
5^a	$t < s < a$
6^a	$t < s = a$
7^a	$t < a < s$
8^a	$a = t < s$

Type	Condition
9^a	$a < t < s$
10^a	$a < t = s$
11^a	$a < s < t$
12^a	$a = s < t$

Figure 1 shows how the vertices in the main apartment are partitioned into the twelve types relative to I^0. For simplicity, we only label the vertices corresponding to elements in $SL_3(F)$.

Every point in X is in the I^a-orbit of some point in the main apartment. This gives a decomposition of X into disjoint sets

$$X = \bigsqcup_{x \in \mathrm{vert}} I^a x K \,/\, K$$

where vert is the set of vertices (s,t).

To pave X by affine spaces, we first divide X into three disjoint sets

$$S = \bigsqcup_{x \text{ type } 1^0,\, 2^0, \text{ or } 3^0} IxK \,/\, K, \qquad T = \bigsqcup_{x \text{ type } 5^0,\, 6^0, \text{ or } 7^0} IxK \,/\, K,$$

and

$$V = X \setminus (S \cup T).$$

The intersection of the I^a-orbit of a vertex v with the set S is denoted by

$$S_v^a = S \cap I^a v K \,/\, K$$

with analogous notation T_v^a for the set T and V_v^a for the set V. In the course of proving the main theorem, we will show that the sets V_v^0, S_v^a and T_v^a are affine spaces and together they give a non-standard paving of X by affine spaces.

We now state the main theorem.

Theorem 1.1. *Let $a = n - m$. The sets $X^\gamma \cap V_v^0$, $X^\gamma \cap S_v^a$, $X^\gamma \cap T_v^a$ are affine spaces that form an affine paving of X^γ as v ranges over the vertices in the main apartment of X.*

2. Understanding I^a-orbits

The sets S and T are defined by I-orbits of vertices, but the main theorem utilizes sets that are defined in part by the I^a-orbits of those vertices. This motivates us to analyze the relationship of I-orbits and I^a-orbits.

To begin, we find a unique matrix coset representative for each point in the I^a-orbit of a vertex (s,t). With this enumeration of points, we can explicitly describe how I and I^a-orbits compare.

2.1. **Enumerating Cosets.**

Consider the I^a-orbit of the vertex $x = (s,t)$,

$$I^a x K \,/\, K = I^a \,\big/\, I^a \cap x K x^{-1}.$$

For an arbitrary matrix in $k \in K$ the conjugate xkx^{-1} is

$$x \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} x^{-1} = \begin{bmatrix} a & b\pi^{-s} & c\pi^{-t} \\ d\pi^s & e & f\pi^{s-t} \\ g\pi^t & h\pi^{t-s} & i \end{bmatrix}.$$

We are interested in the intersection

$$I^a_{(s,t)} = I^a \cap xKx^{-1}.$$

The matrix form of this intersection depends upon the type of x relative to a. For example, when x is type 1^a $(s < a < t)$

$$I^a_{(s,t)} = \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-s} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^t & \mathcal{P}^{t-s} & \mathcal{O}^\times \end{bmatrix}$$

and

$$I^a \Big/ I^a_{(s,t)} = \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-a} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix} \Big/ \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-s} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & \mathcal{O}^\times \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mathcal{P}^{-a} & 0 \\ 0 & 1 & 0 \\ \mathcal{P}^{a+1} & \mathcal{P} & 1 \end{bmatrix} \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-s} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & \mathcal{O}^\times \end{bmatrix} \Big/ \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-s} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & \mathcal{O}^\times \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mathcal{P}^{-a} & 0 \\ 0 & 1 & 0 \\ \mathcal{P}^{a+1} & \mathcal{P} & 1 \end{bmatrix} \Big/ \begin{bmatrix} 1 & \mathcal{P}^{-s} & 0 \\ 0 & 1 & 0 \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & 1 \end{bmatrix}.$$

Then for each $M \in I^a$, we can factor M as

$$\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{bmatrix} \text{ times a matrix in } \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-s} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & \mathcal{O}^\times \end{bmatrix}$$

where

$$\begin{aligned} i &= i_{-a}\pi^{-a} + \cdots + i_{-s-1}\pi^{-s-1} \\ y &= y_{a+1}\pi^{a+1} + \cdots + y_{t-1}\pi^{t-1} \\ z &= z_1\pi^1 + \cdots + z_{t-s-1}\pi^{t-s-1}. \end{aligned}$$

We indicate i, y, and z are such polynomials by writing $i \in \mathcal{P}^{-a} \big/ \mathcal{P}^{-s}$, $y \in \mathcal{P}^{a+1} \big/ \mathcal{P}^{t}$, and $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$.

So each coset in $I^a \Big/ I^a_{(s,t)}$ (when $s < a < t$) can be represented using a triple (i, y, z) by factoring M as above. In fact, each triple represents a unique coset. To see this, suppose the cosets represented by (i, y, z) and (i', y', z') are equivalent, then

$$\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ -y & iy - z & 1 \end{bmatrix} \begin{bmatrix} 1 & i' & 0 \\ 0 & 1 & 0 \\ y' & z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & i' - i & 0 \\ 0 & 1 & 0 \\ y' - y & y(i - i') + (z' - z) & 1 \end{bmatrix} \in I^a_{(s,t)}.$$

This forces $i = i'$ because $i' - i \in \mathcal{P}^{-s}$ but the valuations of i and i' are less than $-s$. Similarly, we have $y = y'$ and $z' = z$. So the set of matrices of the form

$$\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{bmatrix}$$

with $i \in \mathcal{P}^{-a} \big/ \mathcal{P}^{-s}$, $y \in \mathcal{P}^{a+1} \big/ \mathcal{P}^{t}$, and $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$ gives a complete set of unique coset representatives for the elements in $I^a \Big/ I^a_{(s,t)}$ when $s < a < t$. This description also gives a complete enumeration of all the points in the I^a-orbit of a vertex of type 1^a.

A similar analysis can be performed for each vertex type. Using

$$\begin{bmatrix} 1 & i & j \\ x & 1 & k \\ y & z & 1 \end{bmatrix}$$

as a generic coset representative, the following table summarizes how to enumerate, in a standard way, the points in the I^a-orbit of each vertex type.

Type	i	j	k	x	y	z
1^a	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-s}$	0	0	0	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	$\mathcal{P} \,/\, \mathcal{P}^{t-s}$
2^a	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-s}$	0	0	0	0	$\mathcal{P} \,/\, \mathcal{P}^{t-s}$
3^a	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-s}$	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-t}$	0	0	0	$\mathcal{P} \,/\, \mathcal{P}^{t-s}$
4^a	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-s}$	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-t}$	0	0	0	0
5^a	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-s}$	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-t}$	$\mathcal{O} \,/\, \mathcal{P}^{s-t}$	0	0	0
6^a	0	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-t}$	$\mathcal{O} \,/\, \mathcal{P}^{s-t}$	0	0	0
7^a	0	$\mathcal{P}^{-a} \,/\, \mathcal{P}^{-t}$	$\mathcal{O} \,/\, \mathcal{P}^{s-t}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$	0	0
8^a	0	0	$\mathcal{O} \,/\, \mathcal{P}^{s-t}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$	0	0
9^a	0	0	$\mathcal{O} \,/\, \mathcal{P}^{s-t}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	0
10^a	0	0	0	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	0
11^a	0	0	0	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	$\mathcal{P} \,/\, \mathcal{P}^{t-s}$
12^a	0	0	0	0	$\mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	$\mathcal{P} \,/\, \mathcal{P}^{t-s}$

A matrix M that represents a point in the I^a-orbit of a vertex of type r is said to be in standard form if the diagonal entries of M are one and the other entries of M follow the specification for type r in the above table. When referring to a point M, we will also refer to particular values in the standard form of M via the variables $i_M, j_M, k_M, x_M, y_M, z_M$. For example, if M is a point in the I^a-orbit of a vertex of type 1^a, then the $j_M = 0$ and $z_M \in \mathcal{P} \,/\, \mathcal{P}^{t-s}$.

2.2. **Stationary & Non-stationary Points.** Given a point M in the I-orbit of a vertex v, we want to find the I^a-orbit that contains M. If M is in the I^a-orbit of v, we call the point stationary; otherwise the point is called non-stationary. When $a = 0$, all points are stationary and the a-paving is identical to the paving of X by I-orbits. So throughout this section we assume $a > 0$.

Since we only consider the intersection of I^a-orbits with the sets S and T, we only need to compare the I and I^a-orbits for vertices of type 1^0, 2^0, 3^0, 5^0, 6^0, and 7^0.

2.2.1. *v is Type 1^0, 2^0, or 3^0.* This section is devoted to proving the following lemma. Recall that we denote the valuation of the y component of the standard enumeration of the point M in the I-orbit of a vertex by $v(y_M)$.

Lemma 2.1. *Let M be a point in the I-orbit of the vertex $v = (s, t)$.*

(a) *If v is type 2^0 or 3^0 then M is stationary*
(b) *If v is type 1^0 and $v(y_M) > a$ then M is stationary*
(c) *If v is type 1^0 and $v(y_M) \leq a$ then M is in the I^a-orbit of $w = (s-d, t-2d)$ where $d = t - v(y_M)$*

Part (a) & (b): M is stationary if $M \in I \cap I^a$. Since

$$\begin{aligned} I \cap I^a &= \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix} \cap \begin{bmatrix} \mathcal{O}^\times & \mathcal{P}^{-a} & \mathcal{P}^{-a} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix} \end{aligned}$$

we see that any M representing a point in the I-orbit of a vertex of type 2^0, 3^0, or 1^0 with $v(y_M) > a$ is in $I \cap I^a$ and so M is stationary.

Part (c): Since $0 < v(y_M) < t$, we have $d > 0$, which can be thought of as the distance between v and w. To show that M is in the I^a-orbit of w, it suffices to find M' in I^a such that

$$M'w \in MvK.$$

For simplicity we wish to work with matrices rather than matrices modulo scalars. Since $\det(w) = s + t - 3d$ and $\det(v) = s + t$, we replace w with the equivalent element $\pi^d w$ and thus consider

$$\pi^d M'w \in MvK.$$

Rearranging terms yields

$$M^{-1}M' \in \pi^{-d} vKw^{-1}.$$

Since $\pi^{-d}vKw^{-1}$ is the set of matrices of the form

$$\begin{bmatrix} \mathcal{P}^{-d} & \mathcal{P}^{-s} & \mathcal{P}^{d-t} \\ \mathcal{P}^{s-d} & \mathcal{O} & \mathcal{P}^{s-t+d} \\ \mathcal{P}^{t-d} & \mathcal{P}^{t-s} & \mathcal{P}^{d} \end{bmatrix}$$

with determinant a unit and $\det(M^{-1}M')$ is a unit, the condition on M' reduces to

$$M^{-1}M' \in \begin{bmatrix} \mathcal{P}^{-d} & \mathcal{P}^{-s} & \mathcal{P}^{d-t} \\ \mathcal{P}^{s-d} & \mathcal{O} & \mathcal{P}^{s-t+d} \\ \mathcal{P}^{t-d} & \mathcal{P}^{t-s} & \mathcal{P}^{d} \end{bmatrix}.$$

To compute the matrix form of the left side, we need to know the standard form for M', which is determined by w's type relative to I^a. Later, it will be convenient to also know w's type relative to I.

Lemma 2.2. *Suppose that $v = (s,t)$ is type 1^0 and $1 \leq v(y) \leq \min(a, t-1)$. Define $d = t - v(y)$. Then $w = (s-d, t-2d)$ is type 3^a and either type 1^0, 2^0, or 3^0.*

Proof. The inequalities

$$(s-d) < 0 \quad \text{and} \quad (s-d) < (t-2d)$$

imply w is type 1^0, 2^0, or 3^0 and

$$(s-d) < (t-2d) < a$$

implies w is type 3^a.

By assumption, $v(y) \leq \min(a, t-1)$ so $v(y) < t$ and thus $d \geq 1$. Since v is type 1^0 we know $s < 0$. Therefore the inequality $(s-d) < 0$ is clear.

The inequality $(s-d) < (t-2d)$ is equivalent to $s < v(y)$ which follows from $s < 0 < 1 \leq v(y)$. Finally, the inequality $(t-2d) < a$ is equivalent to $2v(y) < a+t$ which follows from $v(y) \leq a$ and $v(y) \leq t-1 < t$. ■

We can now compute $M^{-1}M'$. To simplify notation, we use $i = i_M$, $i' = i_{M'}$, and similar simplifications for the other variables. Note that

$$M = \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{bmatrix}.$$

Because w is type 3^a we will look for $M' \in I^a$ of the form

$$M' = \begin{bmatrix} 1 & i' & j' \\ 0 & 1 & 0 \\ 0 & z' & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} M^{-1}M' &= \begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ -y & iy-z & 1 \end{bmatrix} \begin{bmatrix} 1 & i' & j' \\ 0 & 1 & 0 \\ 0 & z' & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & (i'-i) & j' \\ 0 & 1 & 0 \\ -y & [y(i-i') + (z'-z)] & (1-yj') \end{bmatrix}. \end{aligned}$$

Therefore, we need

$$\begin{bmatrix} 1 & (i'-i) & j' \\ 0 & 1 & 0 \\ -y & [y(i-i') + (z'-z)] & (1-yj') \end{bmatrix} \subset \begin{bmatrix} \mathcal{P}^{-d} & \mathcal{P}^{-s} & \mathcal{P}^{d-t} \\ \mathcal{P}^{s-d} & \mathcal{O} & \mathcal{P}^{s-t+d} \\ \mathcal{P}^{t-d} & \mathcal{P}^{t-s} & \mathcal{P}^{d} \end{bmatrix} \tag{2.1}$$

for M' to represent the point M in the I^a-orbit of w.

From our hypotheses for part (c), we have

$$i \in \mathcal{O} \,/\, \mathcal{P}^{-s}, \quad y \in \mathcal{P} \,/\, \mathcal{P}^{t}, \quad \text{and} \quad z \in \mathcal{P} \,/\, \mathcal{P}^{t-s}$$

with $v(y) \leq \min(a, t-1)$. To determine M', first express z as a polynomial in π

$$z = z_1\pi^1 + z_2\pi^2 + \cdots + z_{t-s-1}\pi^{t-s-1}.$$

and define

$$\lceil z \rceil = z_{t-s-d}\pi^{t-s-d} + \cdots + z_{t-s-1}\pi^{t-s-1}.$$

Let

$$\begin{aligned} z' &= z - \lceil z \rceil & \in & \quad \mathcal{P} \,/\, \mathcal{P}^{t-s-d} \\ i' &= i - \frac{\lceil z \rceil}{y} & \in & \quad \mathcal{O} \,/\, \mathcal{P}^{d-s} \\ j' &= \frac{1}{y} & \in & \quad \mathcal{P}^{d-t} \,/\, \mathcal{P}^{2d-t}. \end{aligned}$$

To verify that $M^{-1}M' \in \pi^{-d}vKw^{-1}$, we check the interesting matrix entries in equation (2.1).

Entry	Check
$(1,2)$	$v(i'-i) = v\left(\frac{\lceil z \rceil}{y}\right) = t-s-d-(t-d) = -s$ so $(i'-i) \in \mathcal{P}^{-s}$
$(1,3)$	$v(j') = -v(y) = d-t$ so $j' \in \mathcal{P}^{d-t}$
$(3,1)$	$v(y) = t-d$ so $-y \in \mathcal{P}^{t-d}$
$(3,2)$	$\begin{aligned} y(i-i')+(z'-z) &= y\left(\frac{\lceil z \rceil}{y}\right) - \lceil z \rceil \\ &= 0 \end{aligned}$
$(3,3)$	$\begin{aligned} (1-yj') &= 1 - y\left(\frac{1}{y}\right) \\ &= 0 \end{aligned}$

Thus the point M in the I-orbit of v is also represented by M' in the I^a-orbit of w. ■

We noted that for a non-stationary point the corresponding w was of type 1^0, 2^0, or 3^0. This shows that when we rearrange elements of S by I^a-orbits, any non-stationary point is in the I^a-orbit of a vertex in S.

Corollary 2.3.

$$S = \bigsqcup_{v \text{ type } 1^0,\ 2^0,\ or\ 3^0} S_v^a$$

Proof. It is clear from the definition of S_v^a that $S_v^a \subset S$.

Let $M \in S$. Then for some v of type 1^0, 2^0, or 3^0

$$M \in IvK \,/\, K.$$

If M is stationary, then $M \in S_v^a$. Otherwise, M is non-stationary and there is a w of type 1^0, 2^0, or 3^0 such that $M \in S_w^a$. ■

2.2.2. *Structure of S_v^a for v of Type 1^0, 2^0, or 3^0.* In the last section, we categorized the points of S as stationary and non-stationary. Given a vertex v of type 1^0, 2^0, or 3^0, we already know which points in S_v^a are stationary. To complete the description of S_v^a, we need to determine the non-stationary points in the set.

When v is type 1^a or 2^a the only points in S_v^a are stationary points because non-stationary move to I^a-orbits of type 3^a vertices. Since only type 1^0 vertices can be type 1^a or 2^a, we can simply apply the stationary condition $v(y) > a$ to the elements in the enumeration of points in the I-orbit of a vertex of type 1^0 to find S_v^a in this case. Therefore, when v is type 1^a or 2^a, we have $s < a \leq t$ and S_v^a is enumerated by

$$\begin{aligned} i &\in \mathcal{O} \,/\, \mathcal{P}^{-s} \\ y &\in \mathcal{P}^{a+1} \,/\, \mathcal{P}^{t} \\ z &\in \mathcal{P} \,/\, \mathcal{P}^{t-s}. \end{aligned}$$

When v is type 3^a, the situation is more complicated because non-stationary points move into I^a-orbits of these vertices. Further complicating the situation is the fact that v can be type 1^0, 2^0, or 3^0 and so the enumeration of the stationary points is different in each case.

To list the stationary points, first suppose that v is type 3^a and either type 1^0 or type 2^0. Then $s < 0 \leq t < a$. If v is type 2^0, then $y = 0$. Otherwise the stationary condition $v(y) > a$ applied to $y \in \mathcal{P} \,/\, \mathcal{P}^t$ forces $y = 0$. So the stationary points in S_v^a are given by

$$\begin{aligned} i &\in \mathcal{O} \,/\, \mathcal{P}^{-s} \\ z &\in \mathcal{P} \,/\, \mathcal{P}^{t-s}. \end{aligned}$$

When v is type 3^a and 3^0, $s < t < 0 < a$ and all the points are stationary

$$\begin{aligned} i &\in \mathcal{O} \,/\, \mathcal{P}^{-s} \\ j &\in \mathcal{O} \,/\, \mathcal{P}^{-t} \\ z &\in \mathcal{P} \,/\, \mathcal{P}^{t-s}. \end{aligned}$$

For each positive integer d, there are non-stationary points in S that contribute to the I^a-orbit of $v = (s, t)$ if v is type 3^a and

(a) $w = (s + d, t + 2d)$ is a type 1^0
(b) there exists $M \in IwK \,/\, K$ so that $v(y) \leq a$ and $d = (t + 2d) - v(y)$ (where $y \equiv y_M$).

Condition (a) locates the vertex of the source of the non-stationary points corresponding to the particular value of d. The second condition identifies non-stationary points that move from vertex w to vertex v.

We can reduce condition (b) by first simplifying the equality in (b) to see that $d = v(y) - t$. Because $0 < d$ we must have $0 < v(y) - t$, or equivalently, $t + 1 \leq v(y)$. With the inequality from (b) we have

$$t + 1 \leq v(y) \leq a. \tag{2.2}$$

Condition (a) translates to the inequality

$$s + d < 0 < t + 2d. \tag{2.3}$$

These two inequalities have slightly different implications depending on the type of v.

When v is type 1^0 we have $s < 0 < t$ so the right side of (2.3) is trivial since $0 < d$. The left side is equivalent to $s + v(y) - t < 0$ or $v(y) < t - s$. Combining this with (2.2) results in

$$t + 1 \leq v(y) \leq \min(a, t - s - 1). \tag{2.4}$$

Otherwise, v is type 2^0 or 3^0 and so $s < t \leq 0$. The left side of (2.2) is trivial since $y \in \mathcal{P} \,/\, \mathcal{P}^t$, so that inequality can be replaced with $1 \leq v(y) \leq a$. Replacing d with $v(y) - t$ changes (2.3) to $s + v(y) - t < 0 < t + 2(v(y) - t)$. The right side is again trivial since $v(y) > 0$ and $t \leq 0$. To satisfy the left side, we need $v(y) < t - s$. Combining this with the modified inequality from (2.2) gives

$$1 \leq v(y) \leq \min(a, t - s - 1). \tag{2.5}$$

Provided y satisfies inequality (2.4) when v is type 1^0 or inequality (2.5) when v is type 2^0 or 3^0, there are non-stationary points in the I-orbit of $w = (s + d, t + 2d)$ that move to the I^a-orbit of v. For all the y of a fixed allowable valuation, the

non-stationary points in S_v^a are enumerated by

$$\begin{array}{rcl} i & \in & \mathcal{O} \Big/ \mathcal{P}^{d-(s+d)} = \mathcal{O} \,/\, \mathcal{P}^{-s} \\ j & \in & \mathcal{P}^{d-(t+2d)} \Big/ \mathcal{P}^{2d-(t+2d)} = \mathcal{P}^{-v(y)} \Big/ \mathcal{P}^{-t} \\ z & \in & \mathcal{P} \Big/ \mathcal{P}^{(t+2d)-(s+d)-d} = \mathcal{P} \,/\, \mathcal{P}^{t-s} \end{array}$$

where $v(j) = -v(y)$ since $j = 1/y$. Thus the non-stationary points correspond to $-1 \geq v(j) \geq -\min(a, t-s-1)$ when v is type 2^0 or 3^0 and $-(t+1) \geq v(j) \geq -\min(a, t-s-1)$ when v is type 1^0. These points fit very nicely with the stationary points at v to give S_v^a enumerated by

$$\begin{array}{rcl} i & \in & \mathcal{O} \,/\, \mathcal{P}^{-s} \\ j & \in & \mathcal{P}^{-x} \,/\, \mathcal{P}^{-t} \\ z & \in & \mathcal{P} \,/\, \mathcal{P}^{t-s} \end{array}$$

where $x = \min(a, t-s-1)$ regardless of whether v is type 1^0, 2^0, or 3^0.

The results from this section are summarized by

Lemma 2.4. *Let $v = (s,t)$ be type 1^0, 2^0, or 3^0 and let $x = \min(a, t-s-1)$. The points in S_v^a are enumerated by*

Type v	i	j	y	z
3^a	$i \in \mathcal{O} \,/\, \mathcal{P}^{-s}$	$j \in \mathcal{P}^{-x} \,/\, \mathcal{P}^{-t}$	$y = 0$	$z \in \mathcal{P} \,/\, \mathcal{P}^{t-s}$
2^a	$i \in \mathcal{O} \,/\, \mathcal{P}^{-s}$	$j = 0$	$y = 0$	$z \in \mathcal{P} \,/\, \mathcal{P}^{t-s}$
1^a	$i \in \mathcal{O} \,/\, \mathcal{P}^{-s}$	$j = 0$	$y \in \mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$	$z \in \mathcal{P} \,/\, \mathcal{P}^{t-s}$

2.2.3. *v is Type 5^0, 6^0, or 7^0.* This section and the next parallel 2.2.1 and 2.2.2. As before, we first prove

Lemma 2.5. *Let M be a point in the I-orbit of the vertex $v = (s,t)$.*

(a) *If v is type 5^0 or 6^0 then M is stationary*
(b) *If v is type 7^0 and $v(x_M) > a$ then M is stationary*
(c) *If v is type 7^0 and $v(x_M) \leq a$ then M is in the I^a-orbit of $w = (s-2d, t-d)$ where $d = s - v(x_M)$*

Part (a) & (b): M is stationary if $M \in I \cap I^a$. From 2.2.1 we know that

$$I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{P}^{a+1} & \mathcal{P} & \mathcal{O}^\times \end{bmatrix}.$$

It is easy to see that any M representing a point in the I-orbit of a vertex of type 5^0, 6^0, or 7^0 with $v(x_M) > a$ is in $I \cap I^a$ and so M is stationary.

Part (c): Since $0 < v(x_M) < s$, we have the distance d between v and w greater than zero. To show that M is in the I^a-orbit of w, it suffices to find M' in I^a such that

$$M'w \in MvK.$$

As in section 2.2.1, we replace w with the equivalent element $\pi^d w$ and consider

$$M^{-1}M' \in \begin{bmatrix} \mathcal{P}^{-d} & \mathcal{P}^{d-s} & \mathcal{P}^{-t} \\ \mathcal{P}^{s-d} & \mathcal{P}^{d} & \mathcal{P}^{s-t} \\ \mathcal{P}^{t-d} & \mathcal{P}^{t-s+d} & \mathcal{O} \end{bmatrix}.$$

To compute the matrix form of the left side, we need to know the standard form for M', which is determined by w's type relative to I^a.

Lemma 2.6. *Suppose that $v = (s,t)$ is type 7^0 and $1 \leq v(x) \leq \min(a, s-1)$. Define $d = s - v(x)$. Then $w = (s-2d, t-d)$ is type 5^a and either type 5^0, 6^0, or 7^0.*

Proof. The inequalities

$$(t-d) < 0 \quad \text{and} \quad (t-d) < (s-2d)$$

imply w is type 5^0, 6^0, or 7^0 and

$$(t-d) < (s-2d) < a$$

implies that w is type 5^a.

By assumption, $v(x) \leq \min(a, s-1)$ so $v(x) < s$ and thus $d \geq 1$. Since v is type 7^0 we know $t < 0$. Therefore the inequality $(t-d) < 0$ is clear.

The inequality $(t-d) < (s-2d)$ is equivalent to $t < v(x)$ which follows from $t < 0 < 1 \leq v(x)$. Finally, the inequality $(s-2d) < a$ is equivalent to $2v(x) < a+s$ which follows from $v(x) \leq a$ and $v(x) \leq s-1 < s$. ■

We now compute $M^{-1}M'$.

Because w is type 5^a,

$$M' = \begin{bmatrix} 1 & i' & j' \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix}$$

and so

$$\begin{aligned} M^{-1}M' &= \begin{bmatrix} 1 & 0 & -j \\ -x & 1 & xj-k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i' & j' \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & i' & (j'-j) \\ -x & (1-xi') & [x(j-j')+(k'-k)] \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, we need

$$\begin{bmatrix} 1 & i' & (j'-j) \\ -x & (1-xi') & [x(j-j')+(k'-k)] \\ 0 & 0 & 1 \end{bmatrix} \subset \begin{bmatrix} \mathcal{P}^{-d} & \mathcal{P}^{d-s} & \mathcal{P}^{-t} \\ \mathcal{P}^{s-d} & \mathcal{P}^{d} & \mathcal{P}^{s-t} \\ \mathcal{P}^{t-d} & \mathcal{P}^{t-s+d} & \mathcal{O} \end{bmatrix} \tag{2.6}$$

for M' to represent the point M in the I^a-orbit of w.

Because M is in the I-orbit of a type 7^0 vertex and non-stationary, we have

$$x \in \mathcal{P} \,/\, \mathcal{P}^s, \quad k \in \mathcal{O} \,/\, \mathcal{P}^{s-t}, \quad \text{and} \quad j \in \mathcal{O} \,/\, \mathcal{P}^{-t}$$

with $v(x) \leq \min(a, s-1)$. To determine M', first express k as a polynomial in π

$$k = k_0 + k_1\pi^1 + k_2\pi^2 + \cdots + k_{s-t-1}\pi^{s-t-1}.$$

and define

$$\lceil k \rceil = k_{s-t-d}\pi^{s-t-d} + \cdots + k_{s-t-1}\pi^{s-t-1}.$$

Let

$$\begin{array}{rclcr} k' & = & k - \lceil k \rceil & \in & \mathcal{O} \,/\, \mathcal{P}^{s-t-d} \\ j' & = & j - \dfrac{\lceil k \rceil}{x} & \in & \mathcal{O} \,/\, \mathcal{P}^{d-t} \\ i' & = & \dfrac{1}{x} & \in & \mathcal{P}^{d-s} \,/\, \mathcal{P}^{2d-s}. \end{array}$$

To verify that $M^{-1}M' \in \pi^{-d}vKw^{-1}$, we check the interesting matrix entries in equation (2.6).

Entry	Check
$(1,2)$	$v(i') = -v(x) = d - s$ so $i' \in \mathcal{P}^{d-s}$
$(1,3)$	$v(j' - j) = v\left(\dfrac{\lceil k \rceil}{x}\right) = s - t - d + (d - s) = -t$
$(2,1)$	$v(-x) = s - d$ so $-x \in \mathcal{P}^{s-d}$
$(2,2)$	$\begin{array}{rcl} (1 - xi') & = & 1 - x\left(\dfrac{1}{x}\right) \\ & = & 0 \end{array}$
$(2,3)$	$\begin{array}{rcl} x(j - j') + (k' - k) & = & x\left(\dfrac{\lceil k \rceil}{x}\right) - \lceil k \rceil \\ & = & 0 \end{array}$

Thus the point M in the I-orbit of v is also represented by M' in the I^a-orbit of w. ■

We showed that for a non-stationary point the corresponding w was of type 5^0, 6^0, or 7^0. This shows that when we rearrange elements of T by I^a-orbits, any non-stationary point is in the I^a-orbit of a vertex in T.

Corollary 2.7.

$$T = \bigsqcup_{v \text{ type } 5^0,\, 6^0,\text{ or } 7^0} T_v^a$$

Proof. It is clear from the definition of T_v^a that $T_v^a \subset T$.

Let $M \in T$. Then for some v of type 5^0, 6^0, or 7^0

$$M \in IvK \,/\, K.$$

If M is stationary, then $M \in T_v^a$. Otherwise, M is non-stationary and there is a w of type 5^0, 6^0, or 7^0 such that $M \in T_w^a$. ■

2.2.4. *Structure of T_v^a for v Type 5^0, 6^0, or 7^0.* In the previous section, we categorized the points of T as stationary and non-stationary. Given a vertex v of type 5^0, 6^0, or 7^0, we already know which points in T_v^a are stationary. To complete the description of T_v^a, we need to determine the non-stationary points in the set.

When v is type 7^a or 6^a the only points in T_v^a are stationary points because non-stationary move to the I^a-orbit of type 5^a vertices. Since only type 7^0 vertices

can be type 7^a or 6^a, we can simply apply the stationary condition $v(x) > a$ to the elements in the enumeration of points in the I-orbit of a vertex of type 7^0 to find T_v^a in this case. Therefore, when v is type 7^a or 6^a, we have $t < a \le s$ and T_v^a is enumerated by

$$\begin{array}{rcl} j & \in & \mathcal{O} \,/\, \mathcal{P}^{-t} \\ k & \in & \mathcal{O} \,/\, \mathcal{P}^{s-t} \\ x & \in & \mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}. \end{array}$$

Now we consider the case when v is type 5^a. To list the stationary points, first suppose that v is type 5^a and either type 7^0 or type 6^0. Then $t < 0 \le s < a$. If v is type 6^0, then $x = 0$. Otherwise the stationary condition $v(x) > a$ applied to $x \in \mathcal{P} \,/\, \mathcal{P}^s$ forces $x = 0$. So the stationary points in T_v^a are given by

$$\begin{array}{rcl} j & \in & \mathcal{O} \,/\, \mathcal{P}^{-t} \\ k & \in & \mathcal{O} \,/\, \mathcal{P}^{s-t}. \end{array}$$

When v is type 5^a and 5^0, $t < s < 0 < a$ and all the points are stationary

$$\begin{array}{rcl} i & \in & \mathcal{O} \,/\, \mathcal{P}^{-s} \\ j & \in & \mathcal{O} \,/\, \mathcal{P}^{-t} \\ k & \in & \mathcal{O} \,/\, \mathcal{P}^{s-t}. \end{array}$$

For each positive integer d, there are non-stationary points in T that contribute to the I^a-orbit of $v = (s,t)$ if v is type 5^a and

(a) $w = (s+2d, t+d)$ is a type 7^0
(b) there exists $M \in IwK \,/\, K$ so that $v(x) > a$ and $d = (s+2d) - v(x)$ (where $x \equiv x_M$).

Condition (a) locates the vertex of the source of the non-stationary points corresponding to the particular value of d. The second condition identifies non-stationary points that move from vertex w to vertex v.

We can reduce condition (b) by first simplifying the equality in (b) to see that $d = v(x) - s$. Because $0 < d$ we must have $0 < v(x) - s$, or equivalently, $s+1 \le v(x)$. With the inequality from (b) we have

$$s + 1 \le v(x) \le a. \tag{2.7}$$

Condition (a) translates to the inequality

$$t + d < 0 < s + 2d. \tag{2.8}$$

These two inequalities have slightly different implications depending on the type of v.

When v is type 7^0 we have $t < 0 < s$ so the right side of (2.8) is trivial since $0 < d$. The left side is equivalent to $t + v(x) - s < 0$ or $v(x) < s - t$. Combining this with (2.7) results in

$$s + 1 \le v(x) \le \min(a, s - t - 1). \tag{2.9}$$

Otherwise, v is type 6^0 or 5^0 and so $t < s \le 0$. The left side of (2.7) is trivial since $x \in \mathcal{P} \,/\, \mathcal{P}^s$, so that inequality can be replaced with $1 \le v(x) \le a$. Replacing d with $v(x) - s$ changes (2.8) to $t + v(x) - s < 0 < s + 2(v(x) - s)$. The right side is

again trivial since $v(x) > 0$ and $s \leq 0$. To satisfy the left side, we need $v(x) < s-t$. Combining this with the modified inequality from (2.7) gives

$$1 \leq v(x) \leq \min(a, s-t-1). \tag{2.10}$$

Provided x satisfies inequality (2.9) when v is type 7^0 or inequality (2.10) when v is type 6^0 or 5^0, there are non-stationary points in the I-orbit of $w = (s+2d, t+d)$ that move to the I^a-orbit of v. For all the x of a fixed allowable valuation, the non-stationary points in T_v^a are enumerated by

$$\begin{aligned} k &\in \mathcal{O} \Big/ \mathcal{P}^{(s+2d)-(t+d)-d} = \mathcal{O} \,/\, \mathcal{P}^{s-t} \\ j &\in \mathcal{O} \Big/ \mathcal{P}^{d-(t+d)} = \mathcal{P} \,/\, \mathcal{P}^{-t} \\ i &\in \mathcal{P}^{d-(s+2d)} \Big/ \mathcal{P}^{2d-(s+2d)} = \mathcal{P}^{-v(x)} \Big/ \mathcal{P}^{-s} \end{aligned}$$

where $v(i) = -v(x)$ since $i = 1/x$. Thus the non-stationary points correspond to $-1 \geq v(i) \geq -\min(a, s-t-1)$ when v is type 6^0 or 5^0 and $-(s+1) \geq v(i) \geq -\min(a, s-t-1)$ when v is type 7^0. These points fit very nicely with the stationary points at v to give T_v^a enumerated by

$$\begin{aligned} k &\in \mathcal{O} \,/\, \mathcal{P}^{s-t} \\ j &\in \mathcal{O} \,/\, \mathcal{P}^{-t} \\ i &\in \mathcal{P}^{-x} \,/\, \mathcal{P}^{-s} \end{aligned}$$

where $x = \min(a, s-t-1)$ regardless of whether v is type 5^0, 6^0, or 7^0.

The results from this section are summarized by

Lemma 2.8. *Let $v = (s,t)$ be type 5^0, 6^0, or 7^0 and let $x = \min(a, s-t-1)$. The points in T_v^a are enumerated by*

Type v	i	j	k	x
5^a	$i \in \mathcal{P}^{-x} \,/\, \mathcal{P}^{-s}$	$j \in \mathcal{O} \,/\, \mathcal{P}^{-t}$	$k \in \mathcal{O} \,/\, \mathcal{P}^{s-t}$	$x = 0$
6^a	$i = 0$	$j \in \mathcal{O} \,/\, \mathcal{P}^{-t}$	$k \in \mathcal{O} \,/\, \mathcal{P}^{s-t}$	$x = 0$
7^a	$i = 0$	$j \in \mathcal{O} \,/\, \mathcal{P}^{-t}$	$k \in \mathcal{O} \,/\, \mathcal{P}^{s-t}$	$x \in \mathcal{P}^{a+1} \,/\, \mathcal{P}^{s}$

3. Proof of the Main Theorem

In this section, we show that the sets $X^\gamma \cap S_v^a$, $X^\gamma \cap T_v^a$, and $X^\gamma \cap V_v^0$ are affine spaces by explicit calculation. Since the standard form of the matrices representing points in X vary depending on the type of v, we proceed case by case through the twelve types. The next section outlines the general approach and the following sections applies the technique to specific regions.

3.1. **Fix Point Conditions.** Let $v = (s,t)$ and M be a point in $IvK \,/\, K = I \,\big/\, I_{(s,t)}$. M is in X^γ if $\gamma M = M$. Since $\gamma \in I_{(s,t)}$, we can replace γM with $\gamma M \gamma^{-1}$ which has the advantage of preserving the standard form of M. Thus M is fixed if and only if $\gamma M \gamma^{-1} \in M I_{(s,t)}$ or equivalently if

$$M^{-1} \gamma M \gamma^{-1} \in I_{(s,t)}. \tag{3.1}$$

Immediately, we see that any vertex in the main apartment is fixed since M is the identity in that case.

The matrix form of (3.1) depends upon the standard form of M and thus v's type. For each of the six vertex types in V, we write M in standard form for that vertex type and compute the left side of (3.1). We then analyze the resulting expression to determine $X^\gamma \cap V_v^0$. In the cases when v is in S_v^a or T_v^a we proceed in a similar fashion except the sets I and $I_{(s,t)}$ are replaced with I^a and $I^a_{(s,t)}$ because S_v^a and T_v^a are affine subsets of the I^a-orbit of v.

3.2. $X^\gamma \cap V_v^0$**.** The set V contains vertices of type 4^0, 8^0, 9^0, 10^0, 11^0, and 12^0. In the following sections, the fixed point condition is applied to the I-orbits of these vertex types and the dimension of the resulting affine space is determined.

3.2.1. v *type* 4^0. The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & -i & -j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \times \quad & \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}\right) & j\left(\frac{u_1}{u_3}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= \quad & \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}-1\right) & j\left(\frac{u_1}{u_3}-1\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\in \quad & \begin{bmatrix} 1 & \mathcal{P}^{-s} & \mathcal{P}^{-t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Thus M is fixed if and only if $v(i) \geq -s-m$ and $v(j) \geq -t-m$.

Since $i \in \mathcal{O} \big/ \mathcal{P}^{-s}$ and $j \in \mathcal{O} \big/ \mathcal{P}^{-t}$ when v is type 4^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(m,-s)+\min(m,-t)$. The variable i contributes $\min(m,-s)$ to the dimension and the variable j contributes $\min(m,-t)$ to the dimension.

3.2.2. v *type* 8^0. The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \quad \times \quad & \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1}\right) & 1 & k\left(\frac{u_2}{u_3}\right) \\ 0 & 0 & 1 \end{bmatrix} \\
= \quad & \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1}-1\right) & 1 & k\left(\frac{u_2}{u_3}-1\right) \\ 0 & 0 & 1 \end{bmatrix} \\
\in \quad & \begin{bmatrix} 1 & 0 & 0 \\ \mathcal{P}^{s} & 1 & \mathcal{P}^{s-t} \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Thus M is fixed if and only if $v(x) \geq s-m$ and $v(k) \geq s-t-n$.

Since $x \in \mathcal{P} \big/ \mathcal{P}^{s}$ and $k \in \mathcal{O} \big/ \mathcal{P}^{s-t}$ when v is type 8^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(m,s-1)+\min(n,s-t)$.

3.2.3. v *type* 9^0. The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ ky-x & 1 & -k \\ -y & 0 & 1 \end{bmatrix} & \times \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1}\right) & 1 & k\left(\frac{u_2}{u_3}\right) \\ y\left(\frac{u_3}{u_1}\right) & 0 & 1 \end{bmatrix} \\
& = \begin{bmatrix} 1 & 0 & 0 \\ ky\left(1-\frac{u_3}{u_1}\right)+x\left(\frac{u_2}{u_1}-1\right) & 1 & k\left(\frac{u_2}{u_3}-1\right) \\ y\left(\frac{u_3}{u_1}-1\right) & 0 & 1 \end{bmatrix} \\
& \in \begin{bmatrix} 1 & 0 & 0 \\ \mathcal{P}^s & 1 & \mathcal{P}^{s-t} \\ \mathcal{P}^t & 0 & 1 \end{bmatrix}.
\end{aligned}$$

From the enumeration of the points in the I-orbit of v, we have $k \in \mathcal{O} \big/ \mathcal{P}^{s-t}$, $x \in \mathcal{P} \big/ \mathcal{P}^s$, and $y \in \mathcal{P} \big/ \mathcal{P}^t$. Incorporating the fixed point conditions, we then have

$$\begin{aligned}
s-t > \quad v(k) & \geq \max(s-t-n, 0) \\
t > \quad v(y) & \geq \max(t-m, 1)
\end{aligned}$$

and

$$x \in \mathcal{P} \big/ \mathcal{P}^s \text{ such that } ky\left(1-\frac{u_3}{u_1}\right)+x\left(\frac{u_2}{u_1}-1\right) \in \mathcal{P}^s.$$

Given a k and y that satisfy the above inequalities, x is partially determined by those values. We write $x = x' + x''$ where x' is the determined portion and x'' is the free portion. Since

$$v\left(ky\left(1-\frac{u_3}{u_1}\right)\right) > 0+1+m$$

and

$$v\left(x\left(\frac{u_2}{u_1}-1\right)\right) \geq 1+m$$

we can always set

$$x' = -ky\frac{\left(1-\frac{u_3}{u_1}\right)}{\left(\frac{u_2}{u_1}-1\right)} \in \mathcal{P} \big/ \mathcal{P}^s$$

and then

$$ky\left(1-\frac{u_3}{u_1}\right)+x'\left(\frac{u_2}{u_1}-1\right) \in \mathcal{P}^s.$$

For any $x'' \in \mathcal{P} \big/ \mathcal{P}^s$ such that

$$v(x'') \geq \max(s-m, 1)$$

we have

$$x''\left(\frac{u_2}{u_1}-1\right) \in \mathcal{P}^s$$

so for $x = x' + x''$,

$$ky\left(1-\frac{u_3}{u_1}\right)+x\left(\frac{u_2}{u_1}-1\right) \in \mathcal{P}^s$$

which satisfies the fixed point condition.

Therefore, when v is type 9^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(n, s-t) + \min(m, t-1) + \min(m, s-1)$.

3.2.4. *v type* 10^0. The fixed point condition (3.1) is

$$\begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -y & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1}\right) & 1 & 0 \\ y\left(\frac{u_3}{u_1}\right) & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1} - 1\right) & 1 & 0 \\ y\left(\frac{u_3}{u_1} - 1\right) & 0 & 1 \end{bmatrix}$$

$$\in \begin{bmatrix} 1 & 0 & 0 \\ \mathcal{P}^s & 1 & 0 \\ \mathcal{P}^t & 0 & 1 \end{bmatrix}.$$

Thus M is fixed if and only if $v(x) \geq s - m$ and $v(y) \geq t - m$.

Since $x \in \mathcal{P} \big/ \mathcal{P}^s$ and $y \in \mathcal{P} \big/ \mathcal{P}^t$ when v is type 10^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(m, s-1) + \min(m, t-1)$.

3.2.5. *v type* 11^0. The fixed point condition (3.1) is

$$\begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ zx - y & -z & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1}\right) & 1 & 0 \\ y\left(\frac{u_3}{u_1}\right) & z\left(\frac{u_3}{u_2}\right) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ x\left(\frac{u_2}{u_1} - 1\right) & 1 & 0 \\ zx\left(1 - \frac{u_2}{u_1}\right) + y\left(\frac{u_3}{u_1} - 1\right) & z\left(\frac{u_3}{u_2} - 1\right) & 1 \end{bmatrix}$$

$$\in \begin{bmatrix} 1 & 0 & 0 \\ \mathcal{P}^s & 1 & 0 \\ \mathcal{P}^t & \mathcal{P}^{t-s} & 1 \end{bmatrix}.$$

From the enumeration of the points in the I-orbit of v, we have $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$, $x \in \mathcal{P} \big/ \mathcal{P}^s$, and $y \in \mathcal{P} \big/ \mathcal{P}^t$. Incorporating the fixed point conditions, we then have

$$\begin{aligned} t - s > \quad v(z) \quad &\geq \max(t - s - n, 1) \\ s > \quad v(x) \quad &\geq \max(s - m, 1) \end{aligned}$$

and

$$y \in \mathcal{P} \big/ \mathcal{P}^t \text{ such that } zx\left(1 - \frac{u_2}{u_1}\right) + y\left(\frac{u_3}{u_1} - 1\right) \in \mathcal{P}^t.$$

Just as x in section 3.2.3 was considered part determined and part free, the entry y is partially determined by the choice of z and x. First, suppose that we choose

suitable z and x that satisfy the fix point condition. Since

$$v\left(zx\left(1-\frac{u_2}{u_1}\right)\right) \geq 1+1+m = 2+m$$

and

$$v\left(y\left(\frac{u_3}{u_1}-1\right)\right) \geq 1+m$$

we can choose a suitable y (in the same manner as we chose x in section 3.2.3) so that $zx\left(1-\frac{u_2}{u_1}\right)+y\left(\frac{u_3}{u_1}-1\right) \in \mathcal{P}^t$ with y'', the free part of y, restricted by

$$t > v(y'') \geq \max(t-m, 1).$$

Therefore, when v is type 11^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(n, t-s-1) + \min(m, s-1) + \min(m, t-1)$.

3.2.6. *v type* 12^0. The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & -z & 1 \end{bmatrix} \quad &\times \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y\left(\frac{u_3}{u_1}\right) & z\left(\frac{u_3}{u_2}\right) & 1 \end{bmatrix} \\
&= \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y\left(\frac{u_3}{u_1}-1\right) & z\left(\frac{u_3}{u_2}-1\right) & 1 \end{bmatrix} \\
&\in \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathcal{P}^t & \mathcal{P}^{t-s} & 1 \end{bmatrix}.
\end{aligned}$$

Thus M is fixed if and only if $v(z) \geq t-s-n$ and $v(y) \geq t-m$. Since $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$ and $y \in \mathcal{P} \big/ \mathcal{P}^t$ when v is type 12^0, the set $X^\gamma \cap V_v^0$ is an affine space of dimension $\min(n, t-s-1) + \min(m, t-1)$.

3.3. $X^\gamma \cap S_v^a$. The set S contains vertices of type 1^a, 2^a, and 3^a. In the following sections, the fixed point conditions are applied to the I^a-orbits of these vertex types and the dimension of the resulting affine space is determined.

Throughout, we reference the enumeration of points in S_v^a described in Lemma 2.4.

3.3.1. *v is type 1^a.* The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ -y & yi-z & 1 \end{bmatrix} \quad & \times \quad \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}\right) & 0 \\ 0 & 1 & 0 \\ y\left(\frac{u_3}{u_1}\right) & z\left(\frac{u_3}{u_2}\right) & 1 \end{bmatrix} \\
& = \quad \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}-1\right) & 0 \\ 0 & 1 & 0 \\ y\left(\frac{u_3}{u_1}-1\right) & yi\left(1-\frac{u_1}{u_2}\right)+z\left(\frac{u_3}{u_2}-1\right) & 1 \end{bmatrix} \\
& \in \quad \begin{bmatrix} 1 & \mathcal{P}^{-s} & 0 \\ 0 & 1 & 0 \\ \mathcal{P}^{t} & \mathcal{P}^{t-s} & 1 \end{bmatrix}.
\end{aligned}$$

Since $i \in \mathcal{O} \,/\, \mathcal{P}^{-s}$ and $y \in \mathcal{P}^{a+1} \,/\, \mathcal{P}^{t}$, to satisfy the fixed point condition we require that

$$\begin{aligned}
-s > \quad & v(i) && \geq \max(-s-m, 0) \\
t > \quad & v(y) && \geq \max(t-m, a+1)
\end{aligned}$$

and

$$z \in \mathcal{P} \,/\, \mathcal{P}^{t-s} \text{ such that } yi\left(1-\frac{u_1}{u_2}\right)+z\left(\frac{u_3}{u_2}-1\right) \in \mathcal{P}^{t-s}.$$

Just as x in section 3.2.3 was considered part determined and part free, the entry z is partially determined by the choice of i and y. First, suppose that we choose suitable i and y that satisfy the fix point condition. Since

$$v\left(yi\left(1-\frac{u_1}{u_2}\right)\right) \geq (a+1)+0+m = n+1$$

and

$$v\left(z\left(\frac{u_3}{u_2}-1\right)\right) \geq 1+n$$

we can choose a suitable z (in the same manner as we chose x in section 3.2.3) so that $yi\left(1-\frac{u_1}{u_2}\right)+z\left(\frac{u_3}{u_2}-1\right) \in \mathcal{P}^{t-s}$ with z'', the free part of z, restricted by

$$t-s > v(z'') \geq \max(t-s-n, 1).$$

Therefore, when v is in S and type 1^a the set $X^\gamma \cap S_v^a$ is an affine space of dimension $\min(m,-s)+\min(m,t-(a+1))+\min(n,t-s-1)$.

3.3.2. *v is type 2^a.* The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ 0 & -z & 1 \end{bmatrix} \quad \times \quad & \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}\right) & 0 \\ 0 & 1 & 0 \\ 0 & z\left(\frac{u_3}{u_2}\right) & 1 \end{bmatrix} \\
= \quad & \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}-1\right) & 0 \\ 0 & 1 & 0 \\ 0 & z\left(\frac{u_3}{u_2}-1\right) & 1 \end{bmatrix} \\
\in \quad & \begin{bmatrix} 1 & \mathcal{P}^{-s} & 0 \\ 0 & 1 & 0 \\ 0 & \mathcal{P}^{t-s} & 1 \end{bmatrix}.
\end{aligned}$$

Thus M is fixed if and only if $v(i) \geq -s-m$ and $v(z) \geq t-s-n$.

Since $i \in \mathcal{O} \big/ \mathcal{P}^{-s}$ and $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$ when v is type 2^a, the set $X^\gamma \cap S_v^a$ is an affine space of dimension $\min(m,-s) + \min(n, t-s-1)$.

3.3.3. *v is type 3^a.* The fixed point condition (3.1) is

$$\begin{aligned}
\begin{bmatrix} 1 & jz-i & -j \\ 0 & 1 & 0 \\ 0 & -z & 1 \end{bmatrix} \quad \times \quad & \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}\right) & j\left(\frac{u_1}{u_3}\right) \\ 0 & 1 & 0 \\ 0 & z\left(\frac{u_3}{u_2}\right) & 1 \end{bmatrix} \\
= \quad & \begin{bmatrix} 1 & jz\left(1-\frac{u_3}{u_2}\right) + i\left(\frac{u_1}{u_2}-1\right) & j\left(\frac{u_1}{u_3}-1\right) \\ 0 & 1 & 0 \\ 0 & z\left(\frac{u_3}{u_2}-1\right) & 1 \end{bmatrix} \\
\in \quad & \begin{bmatrix} 1 & \mathcal{P}^{-s} & \mathcal{P}^{-t} \\ 0 & 1 & 0 \\ 0 & \mathcal{P}^{t-s} & 1 \end{bmatrix}.
\end{aligned}$$

Since $j \in \mathcal{P}^{-x} \big/ \mathcal{P}^{-t}$ (where $x = \min(a, t-s-1)$) and $z \in \mathcal{P} \big/ \mathcal{P}^{t-s}$, to satisfy the fixed point condition we require that

$$\begin{aligned}
-t > \quad & v(j) \quad \geq \max(-t-m, -x) \\
t-s > \quad & v(z) \quad \geq \max(t-s-n, 1)
\end{aligned}$$

and

$$i \in \mathcal{O} \big/ \mathcal{P}^{-s} \text{ such that } jz\left(1-\frac{u_3}{u_2}\right) + i\left(\frac{u_1}{u_2}-1\right) \in \mathcal{P}^{-s}.$$

Just as x in section 3.2.3 was considered part determined and part free, the entry i is partially determined by the choice of j and z. First, suppose that we choose a

suitable j and z that satisfies the fix point condition. Since

$$\begin{aligned} v\left(jz\left(1-\frac{u_3}{u_2}\right)\right) &\geq -\min(a,t-s-1)+1+n \\ &= \max(-a,-(t-s-1))+1+n \\ &\geq -(n-m)+1+n \\ &= m+1 \end{aligned}$$

and

$$v\left(i\left(\frac{u_1}{u_2}-1\right)\right)\geq m$$

we can choose a suitable i (in the same manner as we chose x in section 3.2.3) so that $jz\left(1-\frac{u_3}{u_2}\right)+i\left(\frac{u_1}{u_2}-1\right)\in\mathcal{P}^{-s}$ with i'', the free part of i, restricted by

$$-s>v(i'')\geq\max(-s-m,0).$$

Therefore, when v is in S and type 3^a the set $X^\gamma\cap S_v^a$ is an affine space of dimension $\min(m,-s)+\min(n,t-s-1))+\min(m,a-t,-s-1)$.

3.4. $X^\gamma\cap T$**.** The set T contains vertices of type 5^a, 6^a, and 7^a. In the following sections, the fixed point conditions are applied to the I^a-orbits of these vertex types and the dimension of the resulting affine space is determined.

Throughout, we reference the enumeration of points in T_v^a described in Lemma 2.8.

3.4.1. *v is type 7^a.* The fixed point condition (3.1) is

$$\begin{aligned} \begin{bmatrix} 1 & 0 & -j \\ -x & 1 & xj-k \\ 0 & 0 & 1 \end{bmatrix} &\times \begin{bmatrix} 1 & 0 & j\left(\frac{u_1}{u_3}\right) \\ x\left(\frac{u_2}{u_1}\right) & 1 & k\left(\frac{u_2}{u_3}\right) \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & j\left(\frac{u_1}{u_3}-1\right) \\ x\left(\frac{u_2}{u_1}-1\right) & 1 & xj\left(1-\frac{u_1}{u_3}\right)+k\left(\frac{u_2}{u_3}-1\right) \\ 0 & 0 & 1 \end{bmatrix} \\ &\in \begin{bmatrix} 1 & 0 & \mathcal{P}^{-t} \\ \mathcal{P}^{s} & 1 & \mathcal{P}^{s-t} \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Since $j\in\mathcal{O}\,/\,\mathcal{P}^{-t}$ and $x\in\mathcal{P}^{a+1}\,/\,\mathcal{P}^{s}$, to satisfy the fixed point condition we require that

$$\begin{aligned} -t> \quad & v(j) &\geq \max(-t-m,0) \\ s> \quad & v(x) &\geq \max(s-m,a+1) \end{aligned}$$

and

$$k\in\mathcal{O}\,/\,\mathcal{P}^{s-t}\text{ such that }xj\left(1-\frac{u_1}{u_3}\right)+k\left(\frac{u_2}{u_3}-1\right)\in\mathcal{P}^{s-t}.$$

Just as x in section 3.2.3 was considered part determined and part free, the entry k is partially determined by the choice of j and x. First, suppose that we choose a

suitable j and x that satisfies the fix point condition. Since

$$v\left(xj\left(1-\frac{u_1}{u_3}\right)\right) \geq (a+1)+0+m = n+1$$

and

$$v\left(k\left(\frac{u_2}{u_3}-1\right)\right) \geq 1+n$$

we can choose a suitable k (in the same manner as we chose x in section 3.2.3) so that $xj\left(1-\frac{u_1}{u_3}\right)+k\left(\frac{u_2}{u_3}-1\right) \in \mathcal{P}^{s-t}$ with k'', the free part of k, restricted by

$$s-t > v(k'') \geq \max(s-t-n, 0).$$

Therefore, when v is in T and type 7^a the set $X^\gamma \cap T_v^a$ is an affine space of dimension $\min(m,-t)+\min(m, s-(a+1))+\min(n, s-t)$.

3.4.2. *v is type 6^a.*

$$\begin{bmatrix} 1 & 0 & -j \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & j\left(\frac{u_1}{u_3}\right) \\ 0 & 1 & k\left(\frac{u_2}{u_3}\right) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & j\left(\frac{u_1}{u_3}-1\right) \\ 0 & 1 & k\left(\frac{u_2}{u_3}-1\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\in \begin{bmatrix} 1 & 0 & \mathcal{P}^{-t} \\ 0 & 1 & \mathcal{P}^{s-t} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus M is fixed if and only if $v(j) \geq -t-m$ and $v(k) \geq s-t-n$.

Since $j \in \mathcal{O} \big/ \mathcal{P}^{-t}$ and $k \in \mathcal{O} \big/ \mathcal{P}^{s-t}$ when v is type 6^a, the set $X^\gamma \cap T_v^a$ is an affine space of dimension $\min(m,-t)+\min(n, s-t)$.

3.4.3. *v is type 5^a.* The fixed point condition (3.1) is

$$\begin{bmatrix} 1 & -i & ki-j \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}\right) & j\left(\frac{u_1}{u_3}\right) \\ 0 & 1 & k\left(\frac{u_2}{u_3}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i\left(\frac{u_1}{u_2}-1\right) & ki\left(1-\frac{u_2}{u_3}\right)+j\left(\frac{u_1}{u_3}-1\right) \\ 0 & 1 & k\left(\frac{u_2}{u_3}-1\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\in \begin{bmatrix} 1 & \mathcal{P}^{-s} & \mathcal{P}^{-t} \\ 0 & 1 & \mathcal{P}^{s-t} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $i \in \mathcal{P}^{-x} \big/ \mathcal{P}^{-s}$ (where $x = \min(a, s-t-1)$) and $k \in \mathcal{O} \big/ \mathcal{P}^{s-t}$, to satisfy the fixed point condition we require that

$$\begin{aligned} -s > \quad & v(i) & \geq \max(-s-m, -x) \\ s-t > \quad & v(k) & \geq \max(s-t-n, 0) \end{aligned}$$

and

$$j \in \mathcal{O} \,/\, \mathcal{P}^{-t} \text{ such that } ki\left(1 - \frac{u_2}{u_3}\right) + j\left(\frac{u_1}{u_3} - 1\right) \in \mathcal{P}^{-t}.$$

Just as x in section 3.2.3 was considered part determined and part free, the entry j is partially determined by the choice of k and i. First, suppose that we choose a suitable k and i that satisfies the fix point condition. Since

$$\begin{aligned} v\left(ki\left(1 - \frac{u_2}{u_3}\right)\right) &\geq 0 - \min(a, s-t-1) + n \\ &= \max(-a, -(s-t-1)) + n \\ &\geq -(n-m) + n \\ &= m \end{aligned}$$

and

$$v\left(j\left(\frac{u_1}{u_3} - 1\right)\right) \geq m$$

we can choose a suitable j (in the same manner as we chose x in section 3.2.3) so that $ki\left(1 - \frac{u_2}{u_3}\right) + j\left(\frac{u_1}{u_3} - 1\right) \in \mathcal{P}^{-t}$ with j'', the free part of j, restricted by

$$-t > v(j'') \geq \max(-t-m, 0).$$

Therefore, when v is in T and type 5^a the set $X^\gamma \cap T_v^a$ is an affine space of dimension $\min(m, -t) + \min(n, s-t)) + \min(m, a-s, -t-1)$.

4. Topological Considerations

In the previous sections, we developed a decomposition of X^γ into affine pieces. We now examine the topology of the decomposition to verify it is a paving by affine spaces.

4.1. **Locally Closed.** To show that the affine pieces of X^γ are locally closed, it is enough to show that the sets V_v^0, S_v^a, and T_v^a are locally closed. Every V_v^0 set is simply an I-orbit of a vertex and thus locally closed.

From Lemma 2.4 we know that S_v^a is a closed affine subspace of $I^a vK \,/\, K$, which is locally closed because it is the I^a-orbit of v, and thus S_v^a is locally closed. Similarly, from Lemma 2.8 we can conclude that T_v^a is locally closed.

4.2. **A Filtration by Closed Sets.** In this section, we exhibit a filtration of X by closed subsets that are formed by an increasing union of the affine pieces of X described in the previous sections. Since the affine pieces are in a one-to-one correspondence with the vertices in the main apartment of X, we can describe the filtration by placing an order on those vertices. If $v_i = (s, t)$ is the i-th vertex, let $\mathbb{A}_i$ denote the affine space containing v_i and then

$$\mathbb{A}_0 \subset \mathbb{A}_0 \cup \mathbb{A}_1 \subset \ldots$$

will give the filtration.

The base point $v_0 = (0, 0)$ is a single point and the first closed subset in our filtration. To describe the order on the remaining vertices, we first coarsely group the vertices into increasingly larger sets defined by triangular bounds as in figure 2. The smallest triangle, Δ_1, represents a set of four vertices (three on the boundary and the base point v_0). Triangle Δ_2 is the next largest and it represents the set of six vertices on its boundary plus the four vertices from Δ_1. We define $\Delta_0 = \{v_0\}$.

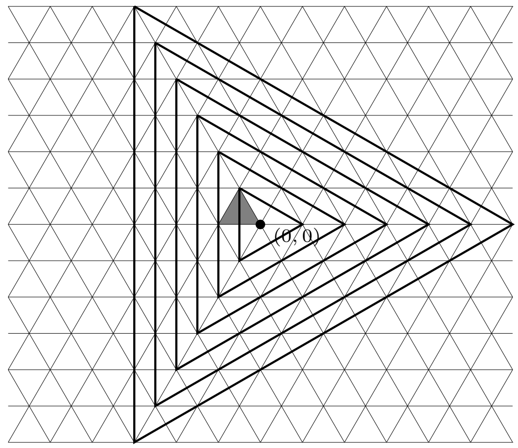

FIGURE 2. Triangular grouping of vertices

The set of points formed by the union of the affine spaces in the a-paving that are indexed by the vertices in Δ_i is equal to the set of points in the union of I-orbits of vertices in Δ_i. To see this, we first refer to Lemma 2.1. In that lemma, we prove that some non-stationary points originate from the I-orbit of a type 1^0 vertex and move to the I^a-orbit of a type 1^a, 2^a, or 3^a vertex. Relative to Δ_i, these non-stationary points move along the upper edge of Δ_i excluding the vertices of the triangle. (See figure 3.) In Lemma 2.5, we prove the other possibility is that

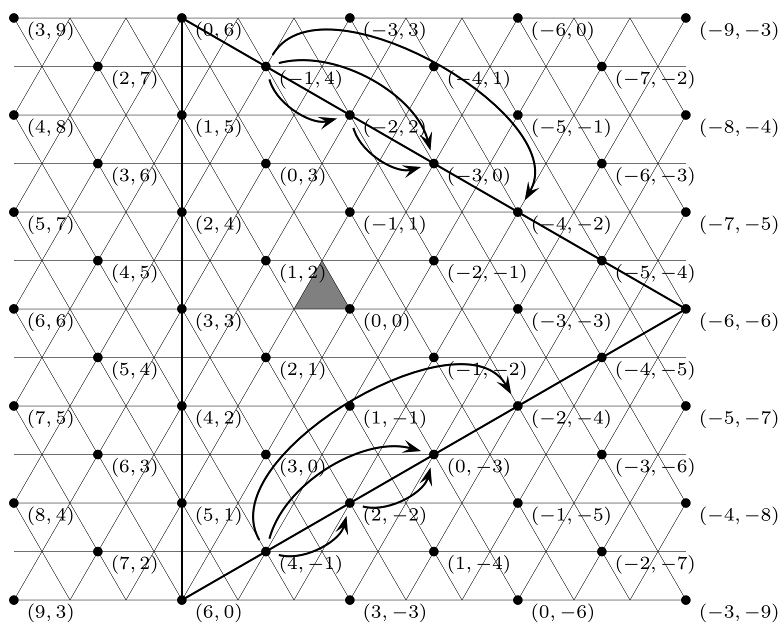

FIGURE 3. The movement of non-stationary points in Δ_5 for $a = 3$

non-stationary points originate from the I-orbit of a type 7^0 vertex and move to

the I^a-orbit of a type 5^a, 6^a, or 7^a vertex. Relative to Δ_i, these non-stationary points move along the lower edge of Δ_i again excluding the vertices of the triangle.

We will now show that if the union of the affine spaces indexed by the vertices $v_0, v_1, \ldots, v_l$ in Δ_{i-1} is closed then there is at least one way to order the vertices $v_{l+1}, v_{l+2}, \ldots, v_k$ on the boundary of Δ_i, such that

$$\mathbb{A}_0 \subset \mathbb{A}_0 \cup \mathbb{A}_1 \subset \cdots \subset \bigcup_{j=0}^{k} \mathbb{A}_j$$

are all closed. Since Δ_0 only contains v_0 and we know that $\mathbb{A}_0$ is closed, we can proceed by induction and order all the vertices of X.

Assume the union of the affine spaces indexed by the vertices in Δ_{i-1} is closed. We will describe a valid order for the vertices on the boundary of Δ_i in three stages. Figure 4 graphically depicts the three stages.

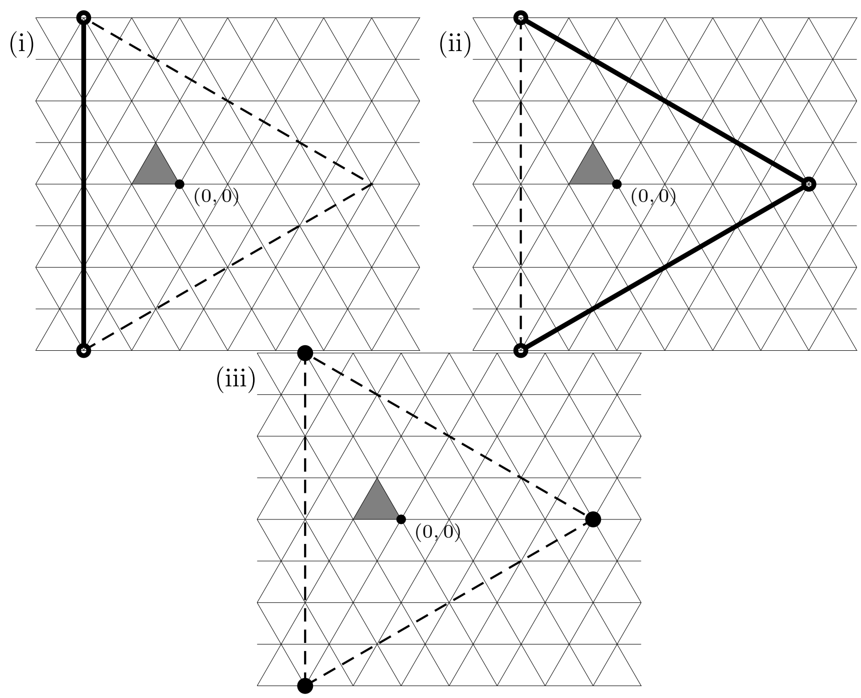

FIGURE 4. Three stages in ordering the vertices of Δ_i

In stage (i), we add the vertices along the vertical edge of Δ_i excluding the two vertices at the ends of this edge. These vertices can be ordered so that the dimensions of the associated affine spaces forms a weakly increasing sequence. To validate the order, we must show that for each v_i in stage (i) the set

$$\bigcup_{j=0}^{i} \mathbb{A}_j$$

is closed. We will use the geometry of minimal galleries to show the union is closed.

Suppose that v_i is a vertex from stage (i). Take a minimal gallery M connecting v to the base alcove (the alcove stabilized by I). We can express M as a word of

simple reflections $M = s_1 \dots s_j$. Each subword of M represents a vertex in X. If every subword of M represents a vertex that is in

$$\bigcup_{j=0}^{i-1} \mathbb{A}_j$$

then

$$\bigcup_{j=0}^{i} \mathbb{A}_j$$

is closed.

Since each affine space associated to a vertex in stage (i) is simply the I-orbit of that vertex, the dimension of the affine space is the length of the minimal gallery connecting the vertex to the alcove stabilized by I. Thus, we are proposing that the vertices in stage (i) can ordered by the length of the minimal gallery associated to the vertex. By considering the possible subwords of a minimal gallery associated to a vertex in stage (i), we see that the vertex represented by the subword is either another stage (i) vertex or a vertex in Δ_{i-1}. See figure 5. Since we order the vertices in stage (i) by the length of the associated minimal gallery, the proposed order is valid.

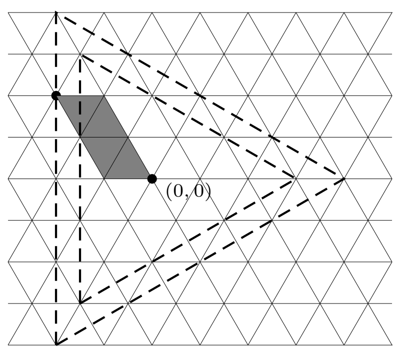

FIGURE 5. Example of a minimal gallery for stage (i)

In stage (ii), we can use the dimension of the I^a-orbit of each vertex to order the vertices. To prove this, it is enough to show that the union Y of the affine spaces associated to Δ_{i-1}, stage (i), and stage (ii) is closed. Then, since the union of the affine spaces associated to Δ_{i-1} and stage (i) is closed, the union of the affine spaces associated to stage (ii) is open in Y. Therefore, we can work in the topological subspace defined by the union of the affine spaces associated to stage (ii). Each of those affine spaces is a subspace of the I^a-orbit of the associated vertex. Hence, if we order the vertices such that the dimension the I^a-orbit of each vertex increases, the order is valid.

To prove that the union of the affine spaces associated to Δ_{i-1}, stage (i), and stage (ii) is closed, we use the fact that the union of the affine spaces associated to the vertices in the convex hull of the Weyl group orbit of a vertex δ is closed. If we let δ be the vertex in stage (ii) that is adjacent to the top most geometric vertex of Δ_i, then from figure 6 we can see the convex hull of the Weyl group orbit of δ is the set of vertices in Δ_{i-1}, stage (i), and stage (ii).

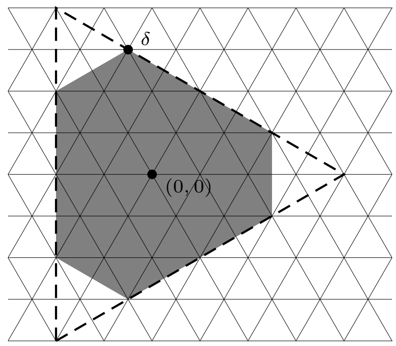

FIGURE 6. The convex hull of the Weyl group orbit of δ

Stage (iii) is the three geometric vertices of Δ_i. Since the union of the affine spaces indexed by the vertices in Δ_i is identical to the union on the I-orbits of those vertices and since the affine spaces associated to the vertices from stage (iii) are I-orbits of those vertices, we can simply order those vertices such that the dimensions of the associated affine spaces forms a weakly increasing sequence.

In figure 7, the vertices of Δ_9 are numbered according to the order we just described. In stage (ii), some I^a-orbits have the same dimension so figure 7 gives one of several valid orderings. Note that the smallest numbers are along the vertical edge, which corresponds to stage (i), and the three largest numbers label the geometric vertices of Δ_9, which corresponds to stage (iii). Further, the vertices in stage (ii) are labeled in order of increasing dimension of the associated I^a-orbit of the vertex (which is equal to the length of a minimal gallery between the vertex and the alcove stabilized by I^a).

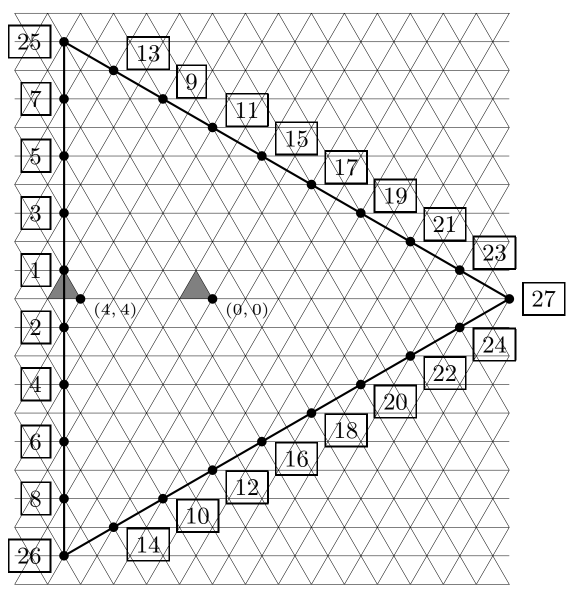

FIGURE 7. Order on the vertices of Δ_9 for $a = 4$

References

[Fan96] C. Kenneth Fan, *Euler characteristic of certain affine flag varieties*, Transform. Groups **1** (1996), 35-39.

[GKM] M. Goresky, R. Kottwitz and R. MacPherson *Purity of equivalued affine Springer fibers*, 2003, arXiv:math.RT/0305141.

[KL88] D. Kazhdan and G. Lusztig, *Fixed point varieties on affine flag manifolds*, Israel J. Math. **62** (1988), 129-168.

[LW] G. Laumon and J.-L. Waldspurger, *Sur le lemme fondamental pour les groupes unitaires: le cas totalement ramifié et homogène*, math.AG/9901114.

[LS91] G. Lusztig and J. M. Smelt, *Fixed point varieties on the space of lattices*, Bull. London Math. Soc. **23** (1991), 213-218.

[Sag00] D. Sage, *The geometry of fixed point varieties on affine flag manifolds*, Trans. Amer. Math. Soc. **352** (2000), 2087-2119.

[Som97] E. Sommers, *A family of affine Weyl group representations*, Transform. Groups **2** (1997), 375-390.

Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, IL 60637

E-mail address: `vincent@math.uchicago.edu`